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Nonlinear Lie algebra and ladder operators for orbital angular momentum

Zhi-xiang Ni

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
Physics Department, Fuyang Teachers College, Fuyang 236032, People's Republic of China†
East China Institute for Theoretical Physics, Shanghai 200237, People's Republic of China

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Abstract. Based on the nonlinear Lie algebra, a new framework is put forward to treat the problem of angular momentum. From this framework, the ladder operators for the orbital angular momentum can be solved as simultaneous eigenvectors of the Lie derivations $\mathrm{ad}(L^2)$ and $\mathrm{ad}(L_z)$ in an operator space. The ladder operators for solvable central potentials can also be found in a united way.

1. Introduction

In recent years, the generalized nonlinear deformations of Lie algebra (including q-deformation Lie algebra [1, 2] as a special case) have attracted a lot of attention. The deformed oscillator algebra as well as the deformed su(2) and su(1, 1) algebras have been discussed by many authors [3–6]. Furthermore, Quesne studied the nonlinear extension of the angular momentum algebra [7]. Recently, we find that the angular momentum itself has naturally a closed nonlinear Lie algebra structure, which is generated by the square of the angular momentum operator, its component and the ladder operator. This nonlinear Lie algebra can be described by two eigenequations in the operator space. Their simultaneous eigenvector operators are the ladder operators for the angular momentum, and the corresponding eigenvalue operators, which determine the differences of eigenvalues of the angular momentum, can be called ladder-value operators. This new framework based on the nonlinear Lie algebra not only gives a new treatment of angular momentum, but also provides a unified way to deal with the eigenvalue problem of the central fields.

2. Nonlinear Lie algebra for orbital angular momentum

For simplicity, we consider here only the orbital angular momentum operator $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ whose Cartesian components L_i, $i = x, y, z$ obey the commutation relations $[L_i, L_j] = \mathrm{i}\epsilon_{ijk} L_k$ (in this paper, we take the natural unit system). It is obvious that

$$[L^2, L_z] = 0. \tag{1}$$

† Address for correspondence.

Thus, the operators L^2 and L_z have simultaneous eigenkets $|l,m\rangle$, corresponding to the eigenvalues μ_l and ν_m respectively, that is

$$\begin{aligned} L^2|l,m\rangle &= \mu_l|l,m\rangle \\ L_z|l,m\rangle &= \nu_m|l,m\rangle. \end{aligned} \tag{2}$$

According to its physical meaning, the ladder operator can map one eigenket of the operators L^2, L_z onto another one, therefore, a ladder operator $X_{s,t}$ for L^2 and L_z should possess the following property:

$$X_{s,t}|l,m\rangle = C_{s,t}|l+s,m+t\rangle \tag{3}$$

where s , $t = -1, 0, +1$, and $C_{s,t}$ are proportionality factors. From equation (3), it is not difficult to show that

$$\begin{aligned} [L^2, X_{s,t}]|l,m\rangle &= (\mu_{l+s} - \mu_l)X_{s,t}|l,m\rangle \\ [L_z, X_{s,t}]|l,m\rangle &= (\nu_{m+t} - \nu_m)X_{s,t}|l,m\rangle. \end{aligned} \tag{4}$$

For convenience, we define two difference functions of the eigenvalues of angular momentum operators as follows:

$$\begin{aligned} F_s(\mu_l) &= \mu_{l+s} - \mu_l \\ G_t(\nu_m) &= \nu_{m+t} - \nu_m. \end{aligned} \tag{5}$$

It is easy to verify, from the completeness of the set $\{|l,m\rangle\}$, that

$$\begin{aligned} [L^2, X_{s,t}] &= X_{s,t}F_s(L^2) \\ [L_z, X_{s,t}] &= X_{s,t}G_t(L_z) \end{aligned} \tag{6}$$

where $F_s(L^2)$ and $G_t(L_z)$ should be called ladder-value operators for the observables L^2 and L_z since their eigenvalues are just the differences of eigenvalues for the same observables. The above equations combined with equation (1) show that the orbital angular operators L^2, L_z and any one of their ladder operators $X_{s,t}$ generate a closed nonlinear Lie algebra. This nonlinear Lie algebra differs from the usual deformed Lie algebras in [4]. The latter has a Cartan subalgebra of dimension one; however, the former has a Cartan subalgebra of dimension two. According to the theory of Lie algebra [8], equations (6) can be viewed as eigenequations of the Lie derivations $\mathrm{ad}(L^2)$ and $\mathrm{ad}(L_z)$, namely,

$$\begin{aligned} \mathrm{ad}(L^2)X &= XF \\ \mathrm{ad}(L_z)X &= XG \end{aligned} \tag{7}$$

but here, F and G take values in an operator space generated by L^2 and L_z. After solving these equations, we can obtain the ladder operators and the corresponding ladder-value operators for L^2 and L_z, then the eigenvalues μ_l and ν_m can be determined.

3. Ladder operators for the orbital angular momentum

In order to solve the eigenvalue equations of $\mathrm{ad}(L^2)$ and $\mathrm{ad}(L_z)$ in the operator space, we will give two useful theorems in the following.

Theorem 1. *If $\boldsymbol{N} = (N_x, N_y, N_z)$ is any vector operator, that is*

$$[L_j, N_k] = \mathrm{i}\varepsilon_{j,k,l}N_l \tag{8}$$

and the operator function

$$\boldsymbol{X}_s = [\tfrac{1}{2}\boldsymbol{N}F_s^2 + \mathrm{i}\boldsymbol{N}\times\boldsymbol{L}F_s - 2(\boldsymbol{N}\cdot\boldsymbol{L})\boldsymbol{L}]C \qquad s = 0, \pm 1 \tag{9}$$

where $F_0 = 0$, $F_{\pm 1} = 1 \pm \sqrt{1 + L^2}$ *and* C *is an arbitrary operator commutative with* L^2*, then the following eigenvalue equation in operator space holds:*

$$\mathrm{ad}(L^2)\boldsymbol{X}_s = \boldsymbol{X}_s \cdot F_s. \tag{10}$$

Proof. For any vector operator $\boldsymbol{N}$ defined in equation (8), we have

$$\begin{aligned} &\mathrm{ad}(L^2)\boldsymbol{N} = 2\boldsymbol{N} + 2\mathrm{i}\boldsymbol{N} \times \boldsymbol{L} \\ &\mathrm{ad}(L^2)\mathrm{i}\boldsymbol{N} \times \boldsymbol{L} = 2 \cdot \boldsymbol{N} \cdot L^2 - 2(\boldsymbol{N} \cdot \boldsymbol{L})\boldsymbol{L} \\ &\mathrm{ad}(L^2)(\boldsymbol{N} \cdot \boldsymbol{L})\boldsymbol{L} = 0. \end{aligned} \tag{11}$$

The above equations imply that the operators $\boldsymbol{N}$ and $\mathrm{i}\boldsymbol{N} \times \boldsymbol{L}$ and $(\boldsymbol{N} \cdot \boldsymbol{L})\boldsymbol{L}$ span a nine-dimensional invariant subspace of $\mathrm{ad}(L^2)$. In this subspace, the eigenvector of $\mathrm{ad}(L^2)$ can be expanded as

$$\boldsymbol{X} = \boldsymbol{N} \cdot C_1 + \mathrm{i}\boldsymbol{N} \times \boldsymbol{L} \cdot C_2 + (\boldsymbol{N} \cdot \boldsymbol{L})\boldsymbol{L} \cdot C_3 \tag{12}$$

where the coefficients C_1, C_2 and C_3 are, in general, operators commutative with L^2. Putting $\boldsymbol{X}$ into equation (10) and using the relations (11), we obtain a set of linear algebraic equations for the coefficients C_j as follows:

$$\begin{pmatrix} 2, & 2L^2, & 0 \\ 2, & 0, & 0 \\ 0, & -2, & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} F. \tag{13}$$

The corresponding secular equation is

$$\begin{vmatrix} 2 - F, & 2L^2, & 0 \\ 2, & -F, & 0 \\ 0, & -2, & -F \end{vmatrix} = 0. \tag{14}$$

Solving equation (14), one obtains three eigenvalues

$$F_0 = 0 \qquad F_{\pm 1} = 1 \pm \sqrt{4L^2 + 1}. \tag{15}$$

From the eigenvalue operators obtained above, we easily get $C_1 : C_2 : C_3 = F^2/2 : F : 1$ for any $F = F_s$, $s = -1, 0, 1$. □

Corollary. If $\boldsymbol{N} = (N_x, N_y, N_z)$ is any vector operator satisfying equation (8) and the scalar product $\boldsymbol{N} \cdot \boldsymbol{L} = 0$, then the operator function

$$\boldsymbol{X}_\sigma(\boldsymbol{N}) = \tfrac{1}{2}\boldsymbol{N}F_\sigma + \mathrm{i}\boldsymbol{N} \times \boldsymbol{L}F_\sigma \qquad \sigma = \pm 1 \tag{16}$$

satisfies the following eigenvalue equation

$$\mathrm{ad}(L^2)\boldsymbol{X}_\sigma(\boldsymbol{N}) = \boldsymbol{X}_\sigma(\boldsymbol{N})F_\sigma. \tag{17}$$

Theorem 2. *If* $\boldsymbol{A} = (A_x, A_y, A_z)$ *and* $\boldsymbol{B} = (B_x, B_y, B_z)$ *are two vector operators and* α, β, γ *are three arbitrary scalar operators, which commutate with* $\boldsymbol{L}$*, then the operator*

$$\boldsymbol{Y} = \boldsymbol{A}\alpha + \boldsymbol{B}\beta + \mathrm{i}\boldsymbol{A} \times \boldsymbol{B}\gamma \tag{18}$$

is a vector operator and the spherical components of $\boldsymbol{Y}$ *satisfies the following eigenvalue equation in operator space*

$$\mathrm{ad}(L_z)Y_t = Y_t t \qquad t = 0, \pm 1 \tag{19}$$

where $Y_0 = Y_z$ *and* $Y_{\pm 1} = Y_x \pm \mathrm{i}Y_y$.

Proof. Since α, β, γ are scaler operators, it is easy to verify that $[L_j, Y_k] = \mathrm{i}\varepsilon_{j,k,l} Y_l$. Thus we have

$$[L_z, Y_0] = 0$$
$$[L_z, Y_{\pm 1}] = \pm Y_{\pm 1}$$

the above result is the same as equation (19). □

Using the above theorems, it is not difficult to solve the equations (7). First, according to theorem 1, we may obtain three eigenvalue operators of $\mathrm{ad}(L^2)$, i.e. the ladder-value operator F_s $(s = 0, \pm 1)$ and the corresponding eigenoperators $\boldsymbol{X}_s$. Since each of the ladder operators $\boldsymbol{X}_s$ has three components, the ladder-value operators F_s are all three-fold degenerate. Then we can solve the eigenvalue equation (7) in each subspace spanned by the ladder operators that belong to the ladder-value operator F_s. According to theorem 2, the ladder operators $\boldsymbol{X}$ are vector operators, the spherical components $X_{s,t}$ of $\boldsymbol{X}_s$ are also the eigen operators of $\mathrm{ad}(L_z)$, corresponding to the eigenvalue operators $G_t = t$ $(t = 0, \pm 1)$. Thus, we obtain altogether nine ladder operators and the corresponding ladder-value operators as listed in the following table:

	$G_0 = 0$	$G_{\pm 1} = \pm 1$
$F_0 = 0$	$X_{0,0} = X_{0,z}$	$X_{0,\pm 1} = X_{0,x} \pm \mathrm{i} X_{0,y}$
$F_{\pm 1} = 1 \pm \sqrt{4L^2 + 1}$	$X_{\pm 1,0} = X_{\pm 1,z}$	$X_{\pm 1,\pm 1} = X_{\pm 1,x} \pm \mathrm{i} X_{\pm 1,y}$

in which the operator $X_{0,0}$ is not a ladder operator in the strict sense.

Since the operator $\boldsymbol{N}$ in theorem 1 is an arbitrary vector operator, there are infinitely many vector ladder operators $\boldsymbol{X}_s$ for a given ladder-value operator F_s of $\mathrm{ad}(L^2)$. For simplicity, we choose $\boldsymbol{N} = \boldsymbol{L}$ and $C = 1/L^2$ for $j = 0$, and $\boldsymbol{N} = \boldsymbol{n}$ $(\boldsymbol{n} = \boldsymbol{r}/r)$ and $C = 1/F_j$ for $j = \sigma = \pm 1$. Thus, using the corollary, we have

$$\begin{aligned} \boldsymbol{X}_0 &= \boldsymbol{L} \\ \boldsymbol{X}_\sigma &= \tfrac{1}{2}\boldsymbol{n} F_\sigma + \mathrm{i}\boldsymbol{n} \times \boldsymbol{L} \end{aligned} \tag{20}$$

or

	$G_0 = 0$	$G_{\pm 1} = \pm 1$
$F_0 = 0$	$X_{0,0} = L_z$	$X_{0,\pm 1} = L_\pm$
$F_\sigma = 1 + \sigma\sqrt{4L^2 + 1}$	$X_{\sigma,0} = n_z(F_\sigma/2 + L_z) + n_- L_+$	$X_{\sigma,\pm 1} = n_\pm(F_\sigma/2 \pm L_z) \mp n_z L_\pm$

According to equation (5), the recurrence formula of μ_l can be obtained directly from the ladder-value operators $F_{\pm 1}(L^2)$, that is

$$\mu_{l\pm 1} = \mu_l + 1 \pm \sqrt{4\mu_l + 1}. \tag{21}$$

The general term corresponding to above equation is

$$\mu_l = \left(\sqrt{\mu_0 + \tfrac{1}{4}} + l\right)^2 - \tfrac{1}{4} \tag{22}$$

where the eigenvalue $\mu_0 = 0$ can be obtained from the condition

$$\boldsymbol{X}_{-1}|0, 0\rangle = 0 \qquad \text{or} \qquad \langle 0, 0|\boldsymbol{X}_{-1}^\dagger \cdot \boldsymbol{X}_{-1}|0, 0\rangle = 0. \tag{23}$$

Thus, we have

$$\mu_l = (l + \tfrac{1}{2})^2 - \tfrac{1}{4} = l(l + 1). \tag{24}$$

This result exactly agrees with the known eigenspectrum of L^2. Similarly, from $G = 0, \pm 1$, one can show that

$$\nu_m = \nu_0 + m. \tag{25}$$

Considering the condition $L_-|l, -l\rangle = 0$, we get $\nu_0 = 0$.

Replacing F_σ in equations (20) with its eigenvalues, we obtain

$$\begin{aligned} X_{\sigma,0} &= n_z(f_\sigma + L_z) + n_- L_+ \\ X_{\sigma,\pm 1} &= n_\pm(f_\sigma \pm L_z) \mp n_z L_\pm \end{aligned} \tag{26}$$

where f_σ equals $l+1$ or $-l$, namely one half of the eigenvalue of the ladder-value operator $F_\sigma(L^2)$. It is easy to verify that the component operators $X_{\sigma,0}$, $X_{\sigma,\sigma}$ and $X_{\sigma,-\sigma}$ are just the same as the ladder operators $B_l^\pm$, $C_l^\pm$ and $D_l^\pm$ constructed in [9], but our results are more natural and concise.

Using equations (26) and the normalization condition, we can calculate the coefficients in the equation (3) as

$$|C_{\sigma,t}|^2 = \langle l, m|X_{\sigma,t}^\dagger X_{\sigma,t}|l, m\rangle = \frac{2l+1}{2l+2\sigma+1} \times \begin{cases} (f_\sigma + m)(f_\sigma + m + 1) & t = 1 \\ (f_\sigma + m)(f_\sigma - m) & t = 0 \\ (f_\sigma - m)(f_\sigma - m + 1) & t = -1. \end{cases} \tag{27}$$

4. Comparing with the linear theory

As we know, the linear Lie algebra associated with the orbital angular momentum is so(3). From the linear so(3) Lie algebra, the ladder operators $L_\pm$ can be derived. However, these two ladder operators $L_\pm$ can only shift the quantum number m in the ket $|l, m\rangle$. To construct the ladder operators which can shift the quantum number l, we need a larger linear Lie algebra so(3, 2) with 10 generators L_j, X_j, Y_j and S [10]. These operators close under the commutation relations

$$\begin{aligned} &[L_j, L_k] = \mathrm{i}\epsilon_{jkl}L_l \qquad [L_j, X_k] = \mathrm{i}\epsilon_{jkl}X_l \\ &[L_j, Y_k] = \mathrm{i}\epsilon_{jkl}Y_l \qquad [L_j, S] = 0 \\ &[X_j, X_k] = -\mathrm{i}\epsilon_{jkl}L_l \qquad [X_j, Y_k] = -\mathrm{i}\delta_{jk}S \\ &[X_j, S] = -\mathrm{i}Y_j \qquad [Y_j, Y_k] = -\mathrm{i}\epsilon_{jkl}L_l \\ &[Y_j, S] = \mathrm{i}X_j. \end{aligned} \tag{28}$$

It is obvious that the framework based on the linear Lie algebra is too complicated to be widely applied.

In contrast, according to our framework, which is based on the nonlinear Lie algebra, all the ladder operators for orbital angular momentum can unitedly be found from equations (7). This set of equations is very concise and can easily be solved by use of the theorems in section 3. Moreover, for any given ladder-value operators, we can construct various ladder operators depending on the given vector operator $\boldsymbol{N}$. This property enables our method to be easily applied to the centrally symmetric fields. In the next section, we will give some typical examples.

5. Applications for the centrally symmetric fields

5.1. Free particle

For a free particle, the Hamilton operator is

$$H = p^2/2. \tag{29}$$

Thus, the operators H, L^2 and L_z have simultaneous eigenkets $\{|k, l, m\rangle\}$, where the parameter $k = \sqrt{2E}$ is the wavevector. Replacing $\boldsymbol{n}$ in equation (20) with the momentum operator $\boldsymbol{p}$, which satisfies the condition $\boldsymbol{p} \cdot \boldsymbol{L} = 0$, we obtain the vector ladder operators $\boldsymbol{X}_\sigma(\boldsymbol{p})$ for the free particle as follows:

$$\boldsymbol{X}_\sigma(\boldsymbol{p}) = \boldsymbol{p}F_\sigma/2 + \mathrm{i}\boldsymbol{p} \times \boldsymbol{L}. \tag{30}$$

Hence, the component ladder operators $X_{\sigma,t}(\boldsymbol{p})$ are

$$\begin{aligned} X_{\sigma,0}(\boldsymbol{p}) &= p_z(F_\sigma/2 + L_z) + p_- L_+ \\ X_{\sigma,\pm 1}(\boldsymbol{p}) &= p_\pm(F_\sigma/2 \pm L_z) \mp p_z L_\pm. \end{aligned} \tag{31}$$

It is not difficult to show that these operators satisfy equation (7) and the commutation relation $[H, X_{\sigma,t}(\boldsymbol{p})] = 0$. The following equation, therefore, holds

$$X_{\sigma,t}(\boldsymbol{p})|k, l, m\rangle = C_{\sigma,t}(\boldsymbol{p})|k, l + \sigma, m + t\rangle \tag{32}$$

where the coefficient $C_{\sigma,t}(\boldsymbol{p}) = k\, C_{\sigma,t}$ ($C_{\sigma,t}$ is defined by equation (27)). Obviously, the above equation gives some useful recurrence relations for the spherical wavefunctions.

5.2. Hydrogen-like atom

In the case of the hydrogen-like atom, we have

$$H = p^2/2 - 1/r. \tag{33}$$

Replacing $\boldsymbol{n}$ in equation (20) with the Runge–Lenz–Pauli vector operator $\boldsymbol{R}$ defined by

$$\boldsymbol{R} = \boldsymbol{n} + \tfrac{1}{2}(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L}) \tag{34}$$

which is a conservative operator and satisfies $\boldsymbol{R} \cdot \boldsymbol{L} = 0$, we obtain the vector ladder operators $\boldsymbol{X}_\sigma(\boldsymbol{R})$ for the hydrogen-like atom

$$\boldsymbol{X}_\sigma(\boldsymbol{R}) = \boldsymbol{R}F_\sigma/2 + \mathrm{i}\boldsymbol{R} \times \boldsymbol{L}. \tag{35}$$

Similarly, the component ladder operators $X_{\sigma,t}(\boldsymbol{R})$ can be found as follows

$$\begin{aligned} X_{\sigma,0}(\boldsymbol{R}) &= R_z(F_\sigma/2 + L_z) + R_- L_+ \\ X_{\sigma,\pm 1}(\boldsymbol{R}) &= R_\pm(F_\sigma/2 \pm L_z) \mp R_z L_\pm. \end{aligned} \tag{36}$$

These ladder operators commute with the Hamilton operator H, and possess the following property:

$$X_{\sigma,t}(\boldsymbol{R})|n, l, m\rangle = C_{\sigma,t}(\boldsymbol{R})|n, l + \sigma, m + t\rangle \tag{37}$$

where the coefficient $C_{\sigma,t}(\boldsymbol{R}) = \sqrt{(n^2 - f_\sigma^2)/n^2}C_{\sigma,t}$, n is the energy quantum number and $\{|n, l, m\rangle\}$ are the simultaneous eigenkets of the operators H, L^2 and L_z.

5.3. Three-dimensional isotropic harmonic oscillator

In this case, we have

$$H = p^2/2 + r^2/2. \tag{38}$$

In the same way, replacing $\boldsymbol{n}$ in $\boldsymbol{X}_\sigma(\boldsymbol{n})$ by the energy ladder operators $\boldsymbol{a}_\pm = (\boldsymbol{r} \mp \mathrm{i}\boldsymbol{p})/\sqrt{2}$, we obtain four vector ladder operators for the three-dimensional isotropic harmonic oscillator

$$\boldsymbol{X}_\sigma(\boldsymbol{a}_\pm) = \boldsymbol{a}_\pm F_\sigma/2 + \mathrm{i}\boldsymbol{a}_\pm \times \boldsymbol{L} \tag{39}$$

and the component ladder operators $X_{\sigma,t}(\boldsymbol{a}_\pm)$ can be obtained as in the previous cases. These ladder operators possess the property

$$X_{\sigma,t}(\boldsymbol{a}_\pm)|n, l, m\rangle = C_{\sigma,t}(\boldsymbol{a}_\pm)|n \pm 1, l + \sigma, m + t\rangle \tag{40}$$

where the coefficient $C_{\sigma,t}(\boldsymbol{a}_\pm) = \sqrt{n + 1 \pm 1 + f_\sigma}\, C_{\sigma,t}$ and n is the energy quantum number.

6. Discussion and conclusion

6.1. Radial ladder operators

Recently, the radial ladder operators are constructed for the hydrogen atom and isotropic harmonic oscillator by use of the factorization in a systematic way [11]. In the following, we will show that their basic radial ladder operators are physically equivalent to the radial components of our vector ladder operators.

For the hydrogen-like atom, we have

$$X_{\sigma,r}(\boldsymbol{R}) = \boldsymbol{n}\cdot\boldsymbol{X}_\sigma(\boldsymbol{R}) = \left(\mathrm{i}p_r - \frac{F_\sigma}{2r} + \frac{2}{F_\sigma}\right)\left(\frac{F_\sigma}{2}\right)^2 \tag{41}$$

where $p_r = (\boldsymbol{n}\cdot\boldsymbol{p}+\boldsymbol{p}\cdot\boldsymbol{n})/2$ is the radial momentum operator. Replacing F_σ in equation (41) with its eigenvalue $2f_\sigma$ and omitting a constant factor, we obtain

$$X_{\sigma,r}(\boldsymbol{R}) = \mathrm{i}p_r - \frac{f_\sigma}{r} + \frac{1}{f_\sigma}. \tag{42}$$

The result is the same as the radial ladder operator $R_l^\pm$ in equation (5.78) in [10]. Obviously, the action of the radial ladder operators $A(l\uparrow)$ and $A(l\downarrow)$ in [11] on the modified radial wavefunctions $\chi_l(r)$ are equivalent to that of the radial components $X_{\pm1,r}(\boldsymbol{R})$ on the radial wavefunctions $R_l(r) = \chi_l(r)/r$, respectively.

As for the three-dimensional isotropic harmonic oscillator, we have

$$X_{\sigma,r}(\boldsymbol{a}_\pm) = \boldsymbol{n}\cdot\boldsymbol{X}_\sigma(\boldsymbol{a}_\pm) = [\mathrm{i}p_r \mp r - F_\sigma/(2r)](F_\sigma/2). \tag{43}$$

Replacing F_σ in equation (43) with its eigenvalue and omitting the constant factor, we obtain

$$X_{\sigma,r}(\boldsymbol{a}_\pm) =' \mathrm{i}p_r \mp r - \frac{f_\sigma}{r}. \tag{44}$$

Similarly, the radial ladder operators $B(l\uparrow, E\uparrow)$, $A(l\downarrow, E\uparrow)$, $A(l\uparrow, E\downarrow)$ and $B(l\downarrow, E\downarrow)$ in [11] are equivalent to the radial components $X_{\pm,r}(\boldsymbol{a}_\pm)$ respectively.

6.2. One-dimensional quantum system

Our method put forward in this paper can also be applied to the one-dimensional quantum system. In the following, we will give a typical example.

For symmetric Poschl–Teller potential, the Hamiltonian of the system is

$$H = p^2/(2m) + V_0\tan^2(kx) \qquad x\in\left(-\frac{\pi}{2k}, \frac{\pi}{2k}\right). \tag{45}$$

Suppose $|n\rangle$ is an eigenket with the eigenvalue E_n, that is

$$H|n\rangle = E_n|n\rangle \tag{46}$$

and $A_\pm$ is the raising or lowering operator, namely

$$A_\pm|n\rangle = C_\pm|n\pm1\rangle \tag{47}$$

then from equations (46) and (47), we have

$$[H, A_\pm]|n\rangle = (E_{n\pm1} - E_n)|n\rangle. \tag{48}$$

For convenience, we define two difference functions of the eigenvalues E_n as follows

$$\Delta_\pm(E_n) = E_{n\pm1} - E_n. \tag{49}$$

From the completeness of the set $|n\rangle$, we have

$$[H, A_\pm] = A_\pm\Delta_\pm(H). \tag{50}$$

Similarly, with the help of the commutation $[x, p] = i\hbar$, we can get the ladder-value operators

$$\Delta_{\pm} = \varepsilon I \pm 2\sqrt{\varepsilon(H + V_0)} \tag{51}$$

where I is the unit operator and $\varepsilon = \hbar^2 k^2/(2m)$. Thus, a recurrence formula of the eigenvalue spectrum can be written as

$$E_{n+1} = E_n + \varepsilon + 2\sqrt{\varepsilon(E_n + V_0)}. \tag{52}$$

The above result is the same as the known result in [6].

It is obvious that our framework based on the nonlinear Lie algebra is more advantageous than that based on the factorization method. The idea we put forward in this paper can also be applied to the other quantum problems.

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